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Real-time approach to tunnelling in open quantum systems: decoherence and anomalous diffusion

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Abstract

Macroscopic quantum tunnelling is described using the master equation for the reduced Wigner function of an open quantum system at zero temperature. Our model consists of a particle trapped in a cubic potential interacting with an environment characterized by dissipative and normal and anomalous diffusion coefficients. A representation based on the energy eigenfunctions of the isolated system, i.e. the system uncoupled to the environment, is used to write the reduced Wigner function, and the master equation becomes simpler in that representation. The energy eigenfunctions computed in a WKB approximation incorporate the tunnelling effect of the isolated system and the effect of the environment is described by an equation that is in many ways similar to a Fokker–Planck equation. Decoherence is easily identified from the master equation and we find that when the decoherence time is much shorter than the tunnelling time the master equation can be approximated by a Kramers-like equation describing thermal activation due to the zero point fluctuations of the quantum environment. The effect of anomalous diffusion can be dealt with perturbatively and its overall effect is to inhibit tunnelling.

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1. Introduction

The observation of quantum tunnelling effects is now possible in some macroscopic quantum variables such as the flux quantum transitions in a superconducting quantum interference device, or the decay of a zero-voltage state in a current-biased Josephson junction [1–4]. Macroscopic quantum systems pertain to the boundary between quantum and classical physics. These systems are modelled by open quantum systems which are characterized by a distinguished subsystem, the ‘system’ for short, interacting with an environment. The environment acts as a source of dissipation and noise for the system and produces quantum

decoherence which generally inhibits tunnelling [5, 6]. There is vast literature on this subject; see [7–9] for comprehensive reviews.

Most work on macroscopic quantum tunnelling is based on imaginary time formalisms such as the Euclidean functional techniques which have been introduced in the classical field of noise-activated escape from a metastable state [10], or the instanton approach introduced for quantum mechanical tunnelling or for vacuum decay in field theory [11–16]. These techniques are specially suited for equilibrium or near equilibrium situations, but are difficult to generalize to non-equilibrium situations.

There are theoretical and practical reasons for a formalism of non-equilibrium macroscopic quantum tunnelling. On the theoretical side, dissipation and decoherence are only truly understood in a dynamical real-time formalism. In the classical context, thermal activation from metastable states is well understood since Kramers [17] in terms of the dynamical Fokker–Planck transport equation, where the roles of dissipation and noise and their inter-relations are known. On the other hand, an open quantum system may be described by a dynamical equation for the reduced density matrix, the so-called master equation, or the equivalent equation for the reduced Wigner function which has many similarities to the Fokker–Planck equation. However, at present no compelling derivation of the tunnelling rate is available in this dynamical framework, which might be compared to the instanton approach for equilibrium systems. Consequently, the effect of dissipation, noise and decoherence on tunnelling and their inter-connections is not yet fully understood. On the practical side, out-of-equilibrium macroscopic quantum tunnelling is becoming necessary to understand arrays of Josephson junctions, or time-dependent traps for cold atoms which are proposed for storing quantum information in future quantum computers [18–21], or to understand first-order phase transitions in cosmology [22, 23].

In recent years, we have considered different scenarios in which metastable quantum systems are described by the master equation for the reduced Wigner function. By using techniques similar to those used for thermal activation processes on metastable states [17, 24] it was possible to compute the effect of the environment on the quantum decay probability. This was used in some semiclassical cosmological scenarios for noise-induced inflation [25] due to the back-reaction of the inflaton field, in the context of stochastic semiclassical gravity [26–30]; see [31, 32] for reviews on this subject. It was also used for bubble nucleation in quantum field theory, where the system was described by the homogeneous mode of the field of bubble size and the environment was played by the inhomogeneous modes of the field [33, 34], and on some simple open quantum systems coupled linearly to a continuum of harmonic oscillators at zero temperature [35]. But in all these problems only the contribution to tunnelling due to activation was considered. One of the purposes of this paper is to clarify when this approximation is valid.

In this paper, we propose a formulation of macroscopic quantum tunnelling using the master equation for the reduced Wigner function in which both the pure quantum tunnelling effect and the environment are taken into account. This is possible by the introduction of a representation of the reduced Wigner function based on the energy eigenfunctions of the isolated system, i.e. the system not coupled to the environment. This representation is useful in a way somewhat analogous to the way the energy representation is useful in the Schrödinger equation. The key to this result is that quantum tunnelling is already encoded in the energy eigenfunctions, which we can compute in a WKB approximation. It is quite remarkable that in this representation the master equation can be solved analytically under certain approximations.

In order to have a working model in a form as simple as possible, but that captures the main physics of the problem, we use constant dissipation and normal and anomalous diffusion

coefficients to describe the effect of the environment. These coefficients can be deduced from microscopic physics: they take constant values when the environment is made by an ohmic distribution of harmonic oscillators weakly coupled in thermal equilibrium at high temperature; but at zero temperature they are time dependent [35–37]. Thus the model studied here may be seen as a toy model at low temperature, generally valid at long time scales only.

Master equations also play an important role in elucidating the emergence of classicality in open quantum systems as a result of their interaction with an environment. In fact, as the master equation gives the quantum evolution of initial states, defined by the reduced Wigner function at some initial time, it has been of great help to study decoherence. In particular, it has been used to clarify the way in which the environment selects a small set of states of the system which are relatively stable by this interaction, the so-called pointer states, whereas the coherent superposition of the remaining states is rapidly destroyed by decoherence [38–42]. Using large-scale numerical simulations the effect of the interaction with the environment on coherent tunnelling has been analysed in the framework of an open quantum system that is classically chaotic: a harmonically driven quartic double well [43, 44]. More recently [45] tunnelling in a simple double-well potential has been numerically simulated using the master equation at high temperature as well as at zero temperature. It is found that at zero temperature tunnelling is inhibited by the environment that produces decoherence, nevertheless at large time scales tunnelling is still possible by an activation-like process due to the zero point fluctuations of the quantum environment. In the model we consider in this paper, which involves a cubic potential, we find a very similar behaviour when the decoherence time is very short.

This paper is organized as follows. In sections 2 and 3 we review the theory of tunnelling in closed systems and introduce the energy representation for Wigner functions. This extended review is necessary both to establish our convention and to recall specific results which are central to the main argument. In section 4 we introduce the environment and write the master equation for the reduced Wigner function of the open quantum system, discuss the different dissipation and diffusion coefficients, and derive the master equation in the energy representation. In section 5 this master equation is explicitly computed and the different time scales of the problem are discussed. In section 6 under the assumption of strong decoherence we analytically solve the master equation, which is reduced to a Kramers-like equation, and then perturbatively compute the effect of anomalous diffusion on tunnelling. Finally, in section 7 we briefly summarize our results. In appendices we provide additional technical details.

2. Tunnelling in quantum mechanics

In this section, we review the WKB method to tunnelling in quantum mechanics. The energy eigenfunctions in the WKB approximation we obtain will play an important role in the energy representation of the Wigner function that will be introduced later.

2.1. The system

We begin with the simple closed quantum mechanical system formed by a particle of mass M in one dimension described by a Hamiltonian

$$H = \frac{p^2}{2M} + U(x), \quad (1)$$

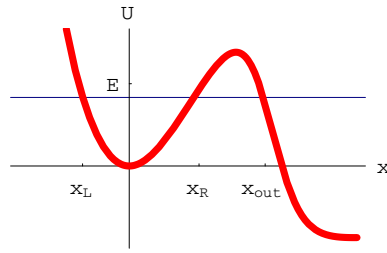


Figure 1. A schematic plot of the potential. For an energy E below the barrier there will be three classical turning points, which are also shown.

(This figure is in colour only in the electronic version)

with a potential U given by

$$U(x) = \frac{1}{2}M\Omega_0^2x^2 - \frac{\lambda}{6}x^3, \quad (2)$$

for small values of the coordinate x . This is a fairly general potential for a tunnelling system, and it is the basic element in the dashboard potential, which is a very good model for a flux trapped in a superconducting quantum interference device (SQUID), or a single Josephson junction biased by a fixed external current [2, 6, 8, 46]. For technical reasons, it is convenient to assume that for large x the potential flattens out and takes the value $U(x) = -U_\infty$, both negative and constant. The tunnelling process ought to be independent of the form of the potential far away from the potential barrier. We present a sketch of this potential in figure 1.

There is one classically stable point at $x = 0$, and one unstable point $x = x_s = 2M\Omega_0^2/\lambda$, corresponding to an energy $\varepsilon_s = 2M^3\Omega_0^6/(3\lambda^2)$. The curvature of the potential is $U''(0) = M\Omega_0^2$ at $x = 0$, and $-U''(0)$ at x_s . The other point at which $U(x) = 0$ is $x = x_{\text{exit}} = (3/2)x_s$. For $x \gg x_{\text{exit}}$ the potential flattens out and is constant.

2.2. The WKB approximation

If we assume that the particle is trapped in the potential well, that is, in its false ground state or false vacuum, the tunnelling probability can be computed in this simple problem in many ways. One of the most efficient methods is the instanton method which reduces to the computation of the ‘bounce solution’. The most attractive aspect of this computation is that it can be easily extended to field theory where the tunnelling probability is then interpreted as the probability per unit time and volume to nucleate a bubble of the true vacuum phase. The rate for quantum tunnelling is $\Gamma_{\text{closed}} = (\Omega_0/2\pi)a_q \exp(-S_B/\hbar)$, where S_B is the action for the ‘bounce’ (or instanton), namely the solution to the classical equations of motion which interpolates between $x = 0$ and $x = x_{\text{exit}}$ in imaginary time $S_B = 2 \int_0^{x_{\text{exit}}} dx \sqrt{2MU(x)}$, and the prefactor $a_q = (120\pi S_B/\hbar)^{1/2}$. Our expression for the potential is so simple that the above integral can be computed explicitly: $S_B/\hbar = 18\varepsilon_s/(5\varepsilon_0)$, where $\varepsilon_0 = \frac{1}{2}\hbar\Omega_0$ is the zero point energy of a harmonic oscillator with frequency Ω_0 .

Here, however, we will concentrate on a real-time approach by expanding the false vacuum state as a linear combination of true eigenstates of the Hamiltonian. To the required accuracy, it is enough to work with the WKB approximations to the true eigenfunctions; see, for instance, [47, 48]. The instanton method reviewed in the previous paragraph can, in fact, be easily justified by this semiclassical approximation. Here we explain in some detail this standard

procedure to obtain the eigenfunctions by matching the WKB solutions in the different regions of the potential. These solutions will play a crucial role in the energy representation for the Wigner functions to be introduced later.

Let $0 < E < \varepsilon_s$ be the energy of the particle in the potential well, and ψ_E the corresponding eigenfunction. The Schrödinger equation is

$$-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} \psi_E + U(x) \psi_E = E \psi_E. \quad (3)$$

Let us define

$$p(x) = \sqrt{2M|U(x) - E|}, \quad (4)$$

and the integral $S(x, y)$ (note the order in the integration limits)

$$S(x, y) = \int_y^x dx' p(x'). \quad (5)$$

The WKB solutions are obtained from these elements. We have to match the WKB solutions in the different regions across the potential function. The details of this calculation are given in appendix A. The WKB solution ψ_E for energies in the range $0 < E < \varepsilon_s$ is given by equation (A.14), where $x_L < x_R < x_{\text{out}}$ are the three classical turning points for the cubic potential (2); see figure 1. The normalization constant K_E in equation (A.14) is obtained by imposing the continuous normalization of the eigenfunctions given in equation (A.16) and it is given in equation (A.23). Of particular relevance is the value of the eigenfunction $\psi_E(x)$ at values $x \gg x_{\text{out}}$. This gives the main contribution to the continuous normalization integral. The value of the eigenfunction at $x \gg x_{\text{out}}$, as computed in appendix A, is

$$\psi_E(x) \sim \sqrt{\frac{2M}{\hbar\pi p_\infty}} \sin\left(\frac{p_\infty x}{\hbar} + \delta_E\right), \quad (6)$$

where the phase δ_E is introduced in equations (A.24) and $p_\infty(E)$ is defined by equation (4) when $x \gg x_{\text{out}}$; see also equation (A.17).

We are interested in the details of the eigenfunctions near the false vacuum state, since we will be dealing with tunnelling from vacuum. Thus, in the rest of this section we give explicitly the values of the normalization constant K_E and the phase shifts δ_E near this vacuum state. Therefore let us impose the Bohr–Sommerfeld quantization condition (A.15) and let E_0 be the corresponding lowest energy, that is, $n = 0$ in equation (A.15). As we will see in the next subsection this defines the false vacuum energy. Expanding the integral in equation (5) around E_0 we find that close to the lowest energy value

$$S(x_R, x_L) \sim \frac{\pi\hbar}{2} - \tau(E - E_0), \quad (7)$$

where τ is defined by

$$\tau = \int_{x_L}^{x_R} dx \sqrt{\frac{2M}{U(x) - E_0}}. \quad (8)$$

Thus $\cos^2(S(x_R, x_L)/\hbar) \sim (\tau^2/\hbar^2)(E - E_0)^2$, and evaluating the right-hand side of (A.26) at E_0 , we conclude that K_E^2 has poles at the complex energies

$$E_\pm = E_0 \pm i\varepsilon, \quad \varepsilon \equiv \frac{\hbar}{4\tau} e^{-2S_0(x_{\text{out}}, x_R)/\hbar}, \quad (9)$$

which is in agreement with the standard result [48]. To simplify the notation let us call $S_0 = S_0(x_{\text{out}}, x_R)$ and $f_0 = f(E_0) + \pi/4$, then we have from equations (A.21) and (A.22) that

the functions $A(E)$ and $B(E)$ for E near E_0 are $A(E) = (\tau/\hbar) \exp(S_0/\hbar)[F_-(E) + F_+(E)]$ and $B(E) = (-i\tau/\hbar) \exp(S_0/\hbar)[F_-(E) - F_+(E)]$, where $F_-(E) = \exp(if_0/\hbar)(E - E_-)$ and $F_+(E) = \exp(-if_0/\hbar)(E - E_+)$. Note that neither A nor B vanish at E_{\pm} . Finally, from equation (A.23) we can write the normalization constant near the false vacuum energy, as

$$K_E^2 = \frac{M}{\pi\hbar\tau} \frac{\varepsilon}{(E - E_0)^2 + \varepsilon^2} = \frac{4M\varepsilon^2}{\pi\hbar^2} \frac{e^{2S_0/\hbar}}{(E - E_-)(E - E_+)}, \tag{10}$$

and from equations (A.24) the phase shifts are

$$e^{i\delta_E} = 2\sqrt{\frac{\varepsilon^2 e^{2S_0/\hbar}}{(E - E_0)^2 + \varepsilon^2}} (A + iB) = e^{if_0/\hbar} \sqrt{\frac{E - E_-}{E - E_+}}. \tag{11}$$

Equations (6), (10) and (11) are the main results of this section. We note, in particular, the poles of the norm and the phase shifts at E_{\pm} near the false vacuum energy. The strong dependence on the energy of these functions near the false ground energy will play an important role in the next sections. We will need, in particular, the phase shift derivatives which are given by

$$\frac{\partial\delta_E}{\partial E} = \frac{-i}{2} \left(\frac{1}{E - E_-} - \frac{1}{E - E_+} \right). \tag{12}$$

2.3. The false vacuum

Before we start with the computation of the tunnelling rate we have to define what we mean by the decaying state; all the wavefunctions we have considered so far are true stationary states and, obviously, show no decay whatsoever. We need to confine initially the particle into the potential well in its lowest energy. To this end, we introduce an auxiliary potential U_{aux} which agrees with U up to x_s (where the true potential reaches its maximum value) and increases thereafter. We may assume that the growth of U_{aux} is as fast as necessary to justify the approximations below; the tunnelling rate is insensitive to the details of U_{aux} beyond x_s . Thus, we define the decaying state ψ_0 as the ground state of a particle confined by U_{aux} [49].

It is obvious from the form of the WKB solutions that ψ_0 agrees with ψ_{E_0} up to x_s , i.e. $\psi_0(x) = \psi_{E_0}(x)$ for $x \leq x_s$, where E_0 is the Bohr–Sommerfeld ground state energy for the auxiliary potential U_{aux} , which corresponds to $n = 0$ in the condition (A.15). Beyond x_s , ψ_0 will decay rapidly to zero, unlike ψ_{E_0} . Like any other wavefunction, ψ_0 admits a development in the complete base of energy eigenfunctions ψ_E , as

$$\psi_0(x) = \int dE C_E \psi_E(x), \tag{13}$$

where due to our normalization the Fourier coefficients are given by

$$C_E = \int dx \psi_E(x) \psi_0(x). \tag{14}$$

To find these coefficients, we observe that $\psi_0(x)$ is a solution to the Schrödinger equation with the auxiliary potential U_{aux}

$$\frac{-\hbar^2}{2M} \frac{\partial^2}{\partial x^2} \psi_0 + U_{\text{aux}}(x) \psi_0 = E_0 \psi_0. \tag{15}$$

Let us add to both sides of this equation the term $[U(x) - U_{\text{aux}}(x)]\psi_0$ and then multiply both sides by $\psi_E(x)$ and integrate to obtain

$$(E - E_0)C_E = - \int_{x_s}^{\infty} dx \psi_E(x) [U_{\text{aux}}(x) - U(x)] \psi_0(x). \tag{16}$$

An important consideration is that $\psi_0(x)$ is a smooth function (as opposed to a distribution), and unlike ψ_{E_0} it is normalizable, so C_E must also be smooth. This means that it is allowable to assume $E \neq E_0$ in equation (16); C_{E_0} can then be found by analytical continuation. To estimate the right-hand side of equation (16), let us introduce, cf equation (4),

$$p_{\text{aux}}(x) = \sqrt{2M|U_{\text{aux}}(x) - E_0|}. \quad (17)$$

To the right of x_s we may use the WKB approximation with the decaying solution into the forbidden region to write

$$\psi_0(x) = \psi_0(x_s) \exp\left(-\frac{1}{\hbar} \int_{x_s}^x p_{\text{aux}}(y) dy\right). \quad (18)$$

On the other hand, $\psi_E(x)$ is given by equation (A.12) in appendix A. If E is close to E_0 , then equation (7) applies, and we may write

$$\begin{aligned} \psi_E(x) \sim 2K_E \left[\frac{\tau}{\hbar} (E - E_0) F_+(x_s, x_R) \exp\left(\frac{1}{\hbar} \int_{x_s}^x p(y) dy\right) \right. \\ \left. + \frac{1}{2} F_-(x_s, x_R) \exp\left(-\frac{1}{\hbar} \int_{x_s}^x p(y) dy\right) \right]. \end{aligned} \quad (19)$$

Substituting the two previous expressions into the right-hand side of equation (16) we see that we have to compute the two following integrals:

$$J_{\pm} = \int_{x_s}^{\infty} dx [U_{\text{aux}}(x) - U(x)] \exp\left(-\frac{1}{\hbar} \int_{x_s}^x [p_{\text{aux}}(y) \pm p(y)] dy\right). \quad (20)$$

The integral, J_- , is dominated by the region near the lower limit, where $p_{\text{aux}}(x)$ is close to $p(x)$ and we can write

$$p_{\text{aux}}(x) - p(x) \sim \frac{p_{\text{aux}}^2(x) - p^2(x)}{2\sqrt{2MU(x_s)}} = \sqrt{\frac{M}{2U(x_s)}} [U_{\text{aux}}(x) - U(x) + E - E_0],$$

from where we obtain

$$J_- = \hbar \sqrt{\frac{2U(x_s)}{M}} - (E - E_0) \int_{x_s}^{\infty} dx \exp\left(-\frac{1}{\hbar} \int_{x_s}^x [p_{\text{aux}}(y) - p(y)] dy\right), \quad (21)$$

where the remaining integral is made negligible by an appropriate choice of U_{aux} . For the other integral, J_+ , we see that the corresponding exponential factor in equation (20) decays faster than the exponential factor of J_- , so that the region which effectively contributes to the integral is narrower. Since the pre-exponential factor vanishes at the lower limit, we find $J_+ \sim 0$. Finally, putting all these pieces together into the right-hand side of equation (16) we get, to leading order,

$$(E - E_0) \left[C_E + 2K_E \psi_0(x_s) \tau \sqrt{\frac{2U(x_s)}{M}} F_+(x_s, x_R) \right] = 0,$$

whose solution, assumed smooth, is

$$C_E = -2K_E \psi_0(x_s) \tau \sqrt{\frac{2U(x_s)}{M}} F_+(x_s, x_R). \quad (22)$$

We note that C_E is independent of the choice of U_{aux} beyond x_s , as it should.

Thus, we have found the false vacuum wavefunction in terms of the energy eigenfunctions of the original problem. The false ground state is a superposition of energy eigenstates which are fine tuned in such a way as to produce destructive interference outside the potential well. Note that C_E , because of the factor K_E in equation (22), peaks near the energy of the false

ground state, and has a strong dependence on the energy near this ground state energy. A good approximation for C_E is given by

$$C_E^2 = \frac{\varepsilon}{\pi} \frac{1}{(E - E_+)(E - E_-)}. \quad (23)$$

2.4. Tunnelling from the false vacuum

Let us now compute the tunnelling rate assuming that the particle is described initially by the false ground state ψ_0 . At time t , we have

$$\psi(x, t) = \int dE e^{-iEt/\hbar} C_E \psi_E(x). \quad (24)$$

The persistence amplitude is

$$\rho(t) = \int dx \psi_0^*(x) \psi(x, t) = \int dE e^{-iEt/\hbar} C_E^2. \quad (25)$$

With the value of C_E^2 given by equation (23) $\rho(0) = 1$. To perform the integration we can close the contour of integration in the complex E plane adding an arc at infinity, whereby we pick up the pole E_- in C_E^2 , therefore $\rho(t)$ goes like

$$\rho(t) \sim \exp \left[\frac{-t}{4\tau} \exp \left(-\frac{2}{\hbar} S_0(x_{\text{out}}, x_R) \right) \right], \quad (26)$$

(no prefactor) provided t is not too large. The tunnelling rate for this closed system, Γ_{closed} , may be defined from the persistence probability $\rho^2(t) \sim \exp(-\Gamma_{\text{closed}}t)$, so that $\Gamma_{\text{closed}} = (1/2\tau) \exp(-2S_0/\hbar)$, which agrees with the result of the bounce solution. Note that if we take the classical lowest energy $E = 0$, then $x_R = x_L = 0$, $x_{\text{out}} = x_{\text{exit}}$, and $S_B = 2S(x_{\text{exit}}, 0)$, but S_0 here is the action corresponding to a particle with false vacuum energy E_0 , which differs from zero, and consequently differs from $S_B/2$. This difference is accounted for by the prefactor a_q in the instanton result. An equivalent way of deriving this result is to estimate the integral by a stationary phase approximation.

3. Wigner function and energy representation

An alternative description of a quantum system is given by the Wigner function in phase space, which is defined by an integral transform of the density matrix [50, 51]. The Wigner function for a system described by a wavefunction $\psi(x)$ is

$$W(x, p) = \int \frac{dy}{2\pi\hbar} e^{ipy/\hbar} \psi \left(x - \frac{y}{2} \right) \psi^* \left(x + \frac{y}{2} \right), \quad (27)$$

where the sign convention is chosen so that a momentum eigenstate $\psi_{p_0}(x) \sim e^{ip_0x/\hbar}/\sqrt{2\pi\hbar}$ becomes $W_{p_0}(x, p) = (1/2\pi\hbar)\delta(p - p_0)$. Moreover, it satisfies

$$\int dp W(x, p) = |\psi(x)|^2, \quad \int dx W(x, p) = \left| \int dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x) \right|^2, \quad (28)$$

and it is normalized so that $\iint dx dp W(x, p) = 1$. Thus the Wigner function is similar in some ways to a distribution function in phase space, and it is real but, unlike a true distribution function, it is not positive defined; this is a feature connected to the quantum nature of the system it describes.

The Schrödinger equation for the wavefunction ψ ,

$$\frac{-\hbar^2}{2M} \frac{\partial^2}{\partial x^2} \psi + U(x)\psi = i\hbar \frac{\partial}{\partial t} \psi, \quad (29)$$

translates into a dynamical equation for the Wigner function, which is easily derived. In fact, by taking the time derivative of (27), using the Schrödinger equation (29), and integrating by parts we have

$$\begin{aligned} \frac{\partial}{\partial t} W(x, p) = & -\frac{i}{\hbar} \int \frac{dy}{2\pi\hbar} e^{ipy/\hbar} \left\{ \left(\frac{-i\hbar p}{M} \right) \frac{\partial}{\partial x} \left[\psi \left(x - \frac{y}{2} \right) \psi^* \left(x + \frac{y}{2} \right) \right] \right. \\ & \left. + \psi \left(x - \frac{y}{2} \right) \left[U \left(x - \frac{y}{2} \right) - U \left(x + \frac{y}{2} \right) \right] \psi^* \left(x + \frac{y}{2} \right) \right\}. \end{aligned}$$

For the cubic potential (2) we have $U(x - y/2) - U(x + y/2) = -M\Omega_0^2 xy + (\lambda/2)x^2 y + (\lambda/24)y^3$ and, noting that $y e^{ipy/\hbar} = -i\hbar \partial_p e^{ipy/\hbar}$ and $y^3 e^{ipy/\hbar} = i\hbar^3 \partial_p^3 e^{ipy/\hbar}$, we get the equation for the Wigner function

$$\frac{\partial}{\partial t} W(x, p) = \left[U'(x) \frac{\partial}{\partial p} - \frac{p}{M} \frac{\partial}{\partial x} + \frac{\lambda}{24} \hbar^2 \frac{\partial^3}{\partial p^3} \right] W(x, p), \quad (30)$$

which may be interpreted as a quantum transport equation. The first two terms on the right-hand side are just the classical Liouville terms for a distribution function, and the term with the three momentum derivatives is responsible for the quantum tunnelling behaviour of the Wigner function in our problem. A theorem by Pawula [52] states that a transport equation should have up to second-order derivatives at most, or else an infinite Kramers–Moyal expansion, for non-negative solutions $W(x, p, t)$ to exist. The above equation for the Wigner function circumvents the implications of the theorem since it need not be everywhere positive. Even if we have an everywhere-positive Gaussian Wigner function at the initial time, the evolution generated by an equation such as equation (30) will not keep it everywhere positive. Thus, here we see the essential role played by the non-positivity of the Wigner function in a genuinely quantum aspect such as tunnelling.

3.1. The energy representation

Given that a wavefunction ψ can be represented in terms of the energy eigenfunctions ψ_E , defined by equation (3), as

$$\psi(x) = \int dE C_E \psi_E(x), \quad (31)$$

we can introduce a corresponding representation for $W(x, p)$ in terms of a base of functions $W_{E_1 E_2}(x, p)$ in phase space defined by

$$W_{E_1 E_2}(x, p) = \int \frac{dy}{2\pi\hbar} e^{ipy/\hbar} \psi_{E_1} \left(x - \frac{y}{2} \right) \psi_{E_2}^* \left(x + \frac{y}{2} \right). \quad (32)$$

Then $W(x, p)$ can be written as

$$W(x, p) = \int dE_1 dE_2 C_{E_1 E_2} W_{E_1 E_2}(x, p), \quad (33)$$

where, in this case, we have $C_{E_1 E_2} = C_{E_1} C_{E_2}^*$. On the other hand, from the definition of $W_{E_1 E_2}(x, p)$ we can write

$$\begin{aligned} \int \frac{dx dp}{\hbar} W_{E_1 E_2}^*(x, p) W_{E'_1 E'_2}(x, p) = & \int \frac{dx dy}{2\pi\hbar^2} \\ & \times \left\{ \psi_{E_1} \left(x - \frac{y}{2} \right) \psi_{E_2} \left(x + \frac{y}{2} \right) \psi_{E'_1} \left(x - \frac{y}{2} \right) \psi_{E'_2} \left(x + \frac{y}{2} \right) \right\}, \end{aligned}$$

where the p integration has been performed. Now if we call $z = x - y/2$, $z' = x + y/2$, then $dx dy = dz dz'$, and

$$\int \frac{dx dp}{\hbar} W_{E_1 E_2}^*(x, p) W_{E_1' E_2'}(x, p) = \frac{1}{2\pi\hbar^2} \delta(E_1 - E_1') \delta(E_2 - E_2'), \quad (34)$$

which gives the orthogonality properties of the functions $W_{E_1 E_2}$. This suggests that any Wigner function may be written in this basis as

$$W(x, p, t) = \int dE_1 dE_2 C_{E_1 E_2}(t) W_{E_1 E_2}(x, p). \quad (35)$$

We call this the energy representation of the Wigner function. In this representation, the master equation or the quantum transport equation (30) is very simple

$$\frac{\partial}{\partial t} C_{E_1 E_2}(t) = \frac{-i}{\hbar} (E_1 - E_2) C_{E_1 E_2}(t), \quad (36)$$

as one can easily verify. One can give an alternative derivation of the tunnelling rate from this equation, by taking the initial condition for the Wigner function which corresponds to the false vacuum.

3.2. Tunnelling in the energy representation: closed system

Let us compute here the tunnelling rate from the false vacuum for our closed quantum system. We assume that our particle at $t = 0$ is trapped into the well of the potential (2) in the false ground state with the energy E_0 , i.e. the ground state of the auxiliary potential U_{aux} introduced in section 2.3. We know from that section that the wavefunction ψ_0 of this state can be expressed in terms of the eigenfunctions ψ_E by equation (13) with the coefficients C_E given by equations (22) and (23). In terms of the reduced Wigner function, which we may call $W_0(x, p)$, this state is easily described in the energy representation (35) by the coefficients $C_{E_1 E_2}(0) = C_{E_1}(0) C_{E_2}^*(0)$, where $C_E(0)$ is given by equation (23). Because the dynamics of the quantum transport equation is trivial in the energy representation (36) the time dependence of the coefficients $C_{E_1 E_2}(t)$ is simply

$$C_{E_1 E_2}(t) = e^{-\frac{i}{\hbar}(E_1 - E_2)t} C_{E_1 E_2}(0). \quad (37)$$

Thus, according to equation (35), the Wigner function at any time is

$$W(x, p, t) = \int dE_1 dE_2 e^{-\frac{i}{\hbar}(E_1 - E_2)t} C_{E_1}(0) C_{E_2}^*(0) W_{E_1 E_2}(x, p). \quad (38)$$

From this we can compute, in particular, the probability of finding the particle at the false vacuum at any time. In terms of the false vacuum Wigner function and the Wigner function of the tunnelling system we may define this probability as

$$\rho^2(t) = 2\pi\hbar \int dx dp W_0(x, p) W(x, p, t). \quad (39)$$

This equation can be used in a closed as well as in an open system. For the closed system of section 2 where the state is described by the wavefunction ψ of equation (24) and the false vacuum is described by the wavefunction ψ_0 of equation (13), the square of the persistence amplitude (25) is given, in fact, by equation (39) when the definition of the Wigner function, i.e. equation (27), is used. For the open system the quantum state is not described by a pure state and, in general, the Wigner function $W(x, p, t)$ can be written as $W = \sum_i p_i W_i$ where p_i is the probability of finding the system in the state ϕ_i and W_i is the Wigner function for the state ϕ_i . The definition (39) leads in this case to $\rho^2(t) = \sum_i p_i |\langle \psi_0 | \phi_i \rangle|^2$, which is indeed the probability of finding the system in the state ψ_0 .

When the energy representation (35) is used equation (39) becomes

$$\rho^2(t) = \int dE_1 dE_2 e^{-\frac{i}{\hbar}(E_1 - E_2)t} C_{E_1}^2(0) C_{E_2}^2(0). \quad (40)$$

To compute $\rho^2(t)$ we shall use the stationary phase approximation. The idea is that the integration paths for E_1 and E_2 may be deformed simultaneously in such a way that the integrand comes to be dominated by Gaussian peaks. For late times it is enough to seek the stationary points of $\frac{i}{\hbar}(E_1 - E_2)$. In principle, we could include $C_{E_1}^2$ and $C_{E_2}^2$ as fast varying components of the integrand, but these functions are really fast varying in the vicinity of E_- and E_+ . Thus, the stationary phase condition leads to $\rho^2 \sim \exp(-2t\varepsilon/\hbar)$ in agreement with the persistence amplitude of equation (26). The tunnelling rate is $\Gamma_{\text{closed}} = 2\varepsilon/\hbar$.

It is often convenient to give the tunnelling rates in terms of an equivalent thermal activation problem under a potential barrier of height ε_s with a certain effective escape temperature T_{esc} . This escape temperature is defined [2] from a given tunnelling rate, Γ , by the equation

$$\Gamma \equiv \frac{1}{2\tau} e^{-\varepsilon_s/k_B T_{\text{esc}}}. \quad (41)$$

For the closed system, either the WKB approximation or the instanton method yields the same T_{esc} with a barrier penetrability Λ , defined by $\Gamma_{\text{closed}} = (1/2\tau) \exp(-\Lambda)$ (which gives $\Lambda = (2/\hbar)S_0$ in the WKB approximation); see equation (D.7) in appendix D.

In the following sections, we will use the energy representation of the Wigner function to compute the tunnelling rate in a more complex problem involving coupling to an environment. The dynamics of the transport equation in the energy representation is simpler than in the phase space representation and the initial condition is given in terms of the coefficients (22) which we have already computed. The task would be more difficult starting from the transport equation in phase space, such as equation (30), since the third derivative term makes the solution of the equation very complicated. One has to resort to methods such as those based on matrix continued fractions in order to compute decay rates from master equations for open quantum systems with third-order derivative terms [52–55].

4. The open quantum system

Now we assume that our system of interest is coupled to an environment. As emphasized by Caldeira and Leggett [6] any quantum macroscopic system can be modelled by an open quantum system by adjusting the coupling of the system and environment variables and by choosing appropriate potentials. One of the main effects of the environment is to induce decoherence into the system which is a basic ingredient into the quantum to classical transition [6, 38–42].

The standard way in which the environment is introduced is to assume that the system is weakly coupled to a continuum set of harmonic oscillators, with a certain frequency distribution. These oscillators represent degrees of freedom to which some suitable variables of the quantum system are coupled. One usually further assumes that the environment is in thermal equilibrium and that the whole system environment is described by the direct product of the density matrices of the system and the environment at the initial time, so that there are no initial system-environment correlations. The macroscopic quantum system is then described by the reduced density matrix, or equivalently, by the reduced Wigner function of the open quantum system. This latter function is defined from the system-environment Wigner function after integration of the environment variables.

In order to have a working model in a form as simple as possible, but that captures the main effect of the environment, we will assume that the reduced Wigner function, which we

still call $W(x, p)$, satisfies the following dynamical equation,

$$\frac{\partial}{\partial t} W(x, p) = \left[U'(x) \frac{\partial}{\partial p} - \frac{p}{M} \frac{\partial}{\partial x} + \frac{\lambda}{24} \hbar^2 \frac{\partial^3}{\partial p^3} + \frac{\partial}{\partial p} \left(\gamma p + \gamma M \sigma^2 \frac{\partial}{\partial p} + \hbar \Delta \frac{\partial}{\partial x} \right) \right] W(x, p), \quad (42)$$

where γ , which has units of inverse time, is the dissipation coefficient, and σ^2 and Δ are, respectively, the normal and anomalous diffusion coefficients. The last three terms of this equation represent the effect of the environment: the first describes the dissipation produced into the system and the other two are the diffusion or noise terms. An interesting limit, the so-called weak dissipation limit, is obtained when $\gamma \rightarrow 0$, so that there is no dissipation, but the coefficient $\gamma \sigma^2$ is kept fixed. We will generally refer to equation (42) as the quantum Kramers equation, or alternatively, as the quantum transport equation. This equation reduces to a classical Fokker–Planck transport equation when $\hbar = 0$: it becomes Kramers' equation [17, 24] for a statistical system coupled to a thermal bath and has the right stationary solutions.

This equation can be derived [36, 56–60] assuming the so-called ohmic distribution for the frequencies of the harmonic oscillators. In the high temperature limit, γ is constant, $\sigma^2 = k_B T$, and $\Delta \propto (k_B T)^{-1}$, where k_B is Boltzmann's constant and T the bath temperature. Thus at high temperature one can generally ignore the anomalous diffusion term. In the low temperature limit, however, the master equation for the reduced Wigner function involves time-dependent dissipation and diffusion coefficients. Typically, the dissipation coefficient $\gamma(t)$ starts with a zero value and after a short transient time, after which the system and the environment become correlated, it reaches a constant value; the normal diffusion coefficient $\sigma^2(t)$ starts also with a zero value, it reaches a maximum, and after the short transient time it undergoes a mild oscillatory behaviour until at time scales $t \gg \Omega^{-1}$ reaches a constant positive asymptotic value. The anomalous diffusion coefficient has a similar qualitative behaviour but its asymptotic large time value is negative and depends on the cut-off frequency. To be specific [35–37, 45], at large time scales the normal diffusion coefficient becomes $\sigma^2 \sim \frac{1}{2} \hbar \Omega_0$, and the anomalous diffusion becomes $\Delta \sim -2\gamma \ln(\Omega_{\text{cut}}/\Omega_0)$, where Ω_{cut} is a suitable cut-off frequency for the ohmic environment. Thus, the vacuum fluctuation of the environment is felt primarily through the anomalous diffusion coefficient that can have a large magnitude. Note that in a macroscopic device such as a single Josephson junction biased by a fixed external current [2, 6, 8, 46] one assumes an ohmic environment just to model the junction resistivity.

Equation (42) is often used to describe the effect of decoherence produced by the diffusion coefficient to study the emergence of classical behaviour in quantum systems; this is a topic of recent interest; see [42] for a review. Of particular relevance to our problem is the study of decoherence in quenched phase transitions [61], and the effect of decoherence in quantum tunnelling in quantum chaotic systems [43, 44], or in a double-well potential [45].

The reduced Wigner function $W(x, p)$ describes the quantum state of the open quantum system, and given a dynamical variable $A(x, p)$ associated with the system its expectation value in that quantum state is defined by

$$\langle A(x, p) \rangle = \int dx dp A(x, p) W(x, p). \quad (43)$$

Then one can easily prove from equation (42) that defining

$$N = \int dx dp W(x, p), \quad \langle E \rangle = \int dx dp \left(\frac{p^2}{2M} + U(x) \right) W(x, p), \quad (44)$$

we have $\dot{N} = 0$ and $\langle \dot{E} \rangle = -\gamma(\langle p^2/M \rangle - N\sigma^2)$. Note that the value of $\sigma^2 \sim \frac{1}{2} \hbar \Omega_0$ for the zero temperature case is reasonable since for the virial theorem the average kinetic energy is half the energy, $\langle p^2/M \rangle = N\sigma^2$, and the averaged energy is conserved $\langle \dot{E} \rangle = 0$.

Further insight into the effect of the different terms of equation (42) can be obtained from the so-called linear entropy $S = 1 - \text{Tr} \rho_r^2$, where ρ_r^2 is the reduced density matrix of the system. This entropy is also a measure of decoherence [62] since for a pure state $\rho_r^2 = \rho_r$ and $S = 0$, whereas for a mixed state $\text{Tr} \rho_r^2 < 1$ and $S > 0$. In position representation $\rho_r = \rho_r(x, y)$ and $\text{Tr} \rho_r^2 = \int dx dy \rho_r(x, y) \rho_r(y, x)$. In terms of the Wigner function, see equation (27), $\rho_r(x, y) = \int dp \exp[-ip(x - y)/\hbar] W[(x + y)/2, p]$. If we now call $X = (x + y)/2$ and $u = x - y$, then $dx dy = dX du$ and we can write

$$\text{Tr} \rho_r^2 = 2\pi\hbar \int dX dp W^2(X, p). \quad (45)$$

We may now compute dS/dt using equation (42). The first three terms are total derivatives and do not contribute to dS/dt . The dissipation term contributes as

$$\frac{dS_{\text{dis}}}{dt} = -2\pi\hbar\gamma \int dX dp W^2(X, p), \quad (46)$$

which is negative and may be understood as reduction of entropy by heat transfer to the environment. The normal diffusion term contributes as

$$\frac{dS_{n.\text{dif}}}{dt} = 4\pi\hbar\gamma M\sigma^2 \int dX dp \left(\frac{\partial W}{\partial p} \right)^2, \quad (47)$$

which is positive and always increases the linear entropy. The anomalous diffusion term, on the other hand, contributes as

$$\frac{dS_{a.\text{dif}}}{dt} = 4\pi\hbar^2\Delta \int dX dp \frac{\partial W}{\partial p} \frac{\partial W}{\partial X}, \quad (48)$$

which has no defined sign. We may infer from here that normal diffusion will always induce decoherence.

4.1. Energy representation of the reduced Wigner function

Let us now use the base of functions in phase space $W_{E_1 E_2}(x, p)$, introduced in equation (32), to represent the reduced Wigner function $W(x, p, t)$ as in equation (35). The previous N and $\langle E \rangle$ have very simple expressions in this representation:

$$N = \int dE C_{EE}(t), \quad \langle E \rangle = \int dE E C_{EE}(t). \quad (49)$$

To check the last equation we note that $\int dx dp [(p^2/2M) + U(x)] W_{E_1 E_2}(x, p) = E_1 \delta(E_1 - E_2)$, which can be easily proved by explicit substitution of the definition of $W_{E_1 E_2}$, and trading powers of p by derivatives with respect to y into expressions (32), and partial integrations.

The quantum transport equation (42) in the energy representation becomes

$$\frac{\partial}{\partial t} C_{E_1 E_2}(t) = \frac{-i}{\hbar} (E_1 - E_2) C_{E_1 E_2}(t) + \int dE'_1 dE'_2 Q_{E_1 E_2, E'_1 E'_2} C_{E'_1 E'_2}(t), \quad (50)$$

where, after one integration by parts,

$$\begin{aligned} Q_{E_1 E_2, E'_1 E'_2} &= -2\pi\hbar^2 \int \frac{dx dp}{\hbar} \left(\frac{\partial}{\partial p} W_{E_1 E_2}^*(x, p) \right) \\ &\quad \times \left(\gamma p + \gamma M\sigma^2 \frac{\partial}{\partial p} + \hbar\Delta \frac{\partial}{\partial x} \right) W_{E'_1 E'_2}(x, p), \end{aligned} \quad (51)$$

which has the contributions from the dissipative, normal diffusion, and anomalous diffusion parts, respectively, as

$$Q_{E_1 E_2, E'_1 E'_2} = Q_{E_1 E_2, E'_1 E'_2}^{(D)} + Q_{E_1 E_2, E'_1 E'_2}^{(N)} + Q_{E_1 E_2, E'_1 E'_2}^{(A)}. \quad (52)$$

From equation (32) it is easy to see that these coefficients can all be written in terms of the following matrix elements:

$$X_{E_1 E_2} = \int dx x \psi_{E_1}(x) \psi_{E_2}(x), \quad (53)$$

$$P_{E_1 E_2} = \frac{\hbar}{i} \int dx \psi_{E_1}(x) \frac{\partial}{\partial x} \psi_{E_2}(x), \quad (54)$$

$$(XP)_{E_1 E_2} = \frac{\hbar}{i} \int dx x \psi_{E_1}(x) \frac{\partial}{\partial x} \psi_{E_2}(x), \quad (55)$$

$$X_{E_1 E_2}^2 = \int dx x^2 \psi_{E_1}(x) \psi_{E_2}(x). \quad (56)$$

Explicitly, we have that

$$\begin{aligned} Q_{E_1 E_2, E'_1 E'_2}^{(D)} &= \frac{-i\gamma}{2\hbar} [(XP)_{E_1 E'_1} \delta(E_2 - E'_2) - P_{E_1 E'_1} X_{E_2 E'_2} \\ &\quad - X_{E_1 E'_1} P_{E_2 E'_2} + (XP)_{E_2 E'_2} \delta(E_1 - E'_1)], \end{aligned} \quad (57)$$

$$Q_{E_1 E_2, E'_1 E'_2}^{(N)} = \frac{\gamma M \sigma^2}{\hbar^2} [2X_{E_1 E'_1} X_{E_2 E'_2} - X_{E_1 E'_1}^2 \delta(E_2 - E'_2) - X_{E_2 E'_2}^2 \delta(E_1 - E'_1)], \quad (58)$$

$$\begin{aligned} Q_{E_1 E_2, E'_1 E'_2}^{(A)} &= \frac{\Delta}{\hbar} [(XP)_{E_1 E'_1} \delta(E_2 - E'_2) - P_{E_1 E'_1} X_{E_2 E'_2} \\ &\quad + X_{E_1 E'_1} P_{E_2 E'_2} - (XP)_{E_2 E'_2} \delta(E_1 - E'_1)]. \end{aligned} \quad (59)$$

Thus, in terms of the Wigner function elements $C_{E_1 E_2}$ the dynamics of the quantum transport equation (50) is simple. Note that the coefficients $Q^{(D)}$ and $Q^{(N)}$ preserve parity, while $Q^{(A)}$ and a coefficient $Q^{(Q)}$ (which corresponds to the pure quantum first term in the equation) change parity; in the sense that the symmetric and antisymmetric parts of $C_{E_1 E_2}$ are independently preserved or interchanged when contracted to these terms.

Equation (50) resembles a similar equation when a Floquet basis of states is used [63–66], which is very useful when the Hamiltonian of the system is periodic in time. The Floquet basis is discrete in such a case and a numerical evaluation of the corresponding matrix elements (53)–(56) can be performed; see, for instance, [43, 44] for a recent application. It is remarkable that in our case approximated analytic expressions for these matrix elements can be found.

4.2. Some properties of the matrix elements

The matrix elements (53)–(56) have a clear physical interpretation and several relations can be derived among them. Note that $X_{E_1 E_2}$ is the matrix element of the position operator X in the energy representation. Since $X\psi_E(x) = x\psi_E(x)$, we must have $\int dE_1 X_{E E_1} \psi_{E_1}(x) = x\psi_E(x)$.

On the other hand, $P_{E_1 E_2}$ is the matrix element for the momentum operator. The canonical commutation relation $[P, X] = -i\hbar$ implies $[H, X] = (-i\hbar/M)P$, and taking matrix elements on both sides we have

$$(E_1 - E_2)X_{E_1 E_2} = -\frac{i\hbar}{M}P_{E_1 E_2}. \quad (60)$$

Also, $X_{E_1 E_2}^2$ is the matrix element of X^2 , therefore

$$X_{E_1 E_2}^2 = \int dE X_{E_1 E} X_{E E_2}. \quad (61)$$

On the other hand, $(XP)_{E_1E_2}$ is the matrix element of XP , consequently $[(XP)_{E_2E_1}]^* = -(XP)_{E_2E_1}$ corresponds to PX , and $(XP)_{E_1E_2} + (XP)_{E_2E_1} = [X, P]_{E_1E_2} = i\hbar\delta(E_1 - E_2)$. Also $(XP)_{E_1E_2} - (XP)_{E_2E_1} = (iM/\hbar)(E_1 - E_2)X_{E_1E_2}^2$, where the commutator $[H, X^2]$ has been used in the last step, therefore

$$(XP)_{E_1E_2} = \frac{iM}{2\hbar}(E_1 - E_2)X_{E_1E_2}^2 + \frac{i\hbar}{2}\delta(E_1 - E_2). \quad (62)$$

We also have that $(XP)_{E_1E_2} = \int dE X_{E_1E} P_{EE_2}$. One may check, for consistency, that these relations imply $\dot{N} = 0$. In appendix B a test of the quantum transport equation in the energy representation (and of the above matrix element properties) is given by checking that a stationary solution with a thermal spectrum is, indeed, a solution in the high temperature limit.

4.3. Computing the matrix elements

The matrix elements contain singular parts coming from the integrals over the unbound region beyond x_s . These singular parts are easy to compute, since far enough the wavefunctions assume the simple form (6). When performing the calculation of the singular parts of the matrix elements we will use that when $\bar{x} \rightarrow \infty$, we have the identities

$$\frac{\sin(p\bar{x}/\hbar)}{\pi p} \rightarrow \delta(p), \quad \frac{\cos(p\bar{x}/\hbar)}{p} \rightarrow 0, \quad (63)$$

which can be easily checked by taking the Fourier transforms of these functions with respect to p .

The computation of the singular parts of the matrix elements (53)–(56) may be reduced to the evaluation of three basic integrals. These integrals are

$$A_{A,S}(p_1, p_2) = \int dx \sin[(p_1 \mp p_2)x/\hbar + \delta_1 \mp \delta_2], \quad (64)$$

and

$$B(p_1, p_2) = \int dx \sin(p_1x/\hbar + \delta_1) \sin(p_2x/\hbar + \delta_2), \quad (65)$$

where, for simplicity, we have written $p_i \equiv p_\infty(E_i)$ and $\delta_i \equiv \delta(E_i)$ ($i = 1, 2$). The matrix element $X_{E_1E_2}$ is

$$\begin{aligned} X_{E_1E_2} &\sim \frac{2M}{\hbar\pi\sqrt{p_1p_2}} \int dx x \sin(p_1x/\hbar + \delta_1) \sin(p_2x/\hbar + \delta_2) \\ &= \frac{M}{\pi\sqrt{p_1p_2}} \left[-\frac{\partial A}{\partial p_1} - \frac{\partial \tilde{A}}{\partial p_2} - \left(\frac{\partial \delta_1}{\partial p_1} + \frac{\partial \delta_2}{\partial p_2} \right) B \right], \end{aligned} \quad (66)$$

where $A \equiv (A_S - A_A)/2$ and $\tilde{A} \equiv (A_S + A_A)/2$. The matrix element $X_{E_1E_2}^2$ is

$$\begin{aligned} X_{E_1E_2}^2 &\sim \frac{2M}{\hbar\pi\sqrt{p_1p_2}} \int dx x^2 \sin(p_1x/\hbar + \delta_1) \sin(p_2x/\hbar + \delta_2) \\ &= \frac{2M}{\pi\sqrt{p_1p_2}} \left[-\frac{\partial C}{\partial p_1} - \left(\frac{\partial \delta_1}{\partial p_1} \right) \tilde{D} \right], \end{aligned} \quad (67)$$

where it is easy to show that $C = (\partial B/\partial p_1) - (\partial \delta_1/\partial p_1)A$, and that $\tilde{D} = -(\partial A/\partial p_1) - (\partial \delta_1/\partial p_1)B$. The matrix element $P_{E_1E_2}$ is

$$P_{E_1E_2} \sim \frac{-iM}{\hbar\pi\sqrt{p_1p_2}}(p_1 + p_2)\tilde{A}, \quad (68)$$

which, according to the relations among matrix elements derived in the previous subsection, is related to $X_{E_1E_2}$ by equation (60). The remaining matrix element $(XP)_{E_1E_2}$, on the other hand, can be computed from the element $X_{E_1E_2}^2$ according to equation (62)

4.3.1. *The integrals $A(p_1, p_2)$ and $B(p_1, p_2)$.* Thus, we are finally left with the computation of the integrals (64) and (65). The integral $B(p_1, p_2)$ of equation (65) is dominated by its upper limit \bar{x}

$$\begin{aligned} B(p_1, p_2) &\sim \frac{1}{2} \int^{\bar{x}} dx \cos[(p_1 - p_2)x/\hbar + \delta_1 - \delta_2] \\ &\sim \frac{1}{2(p_1 - p_2)} \sin[(p_1 - p_2)\bar{x}/\hbar + \delta_1 - \delta_2] \\ &\rightarrow \frac{\pi\hbar}{2} \delta(p_1 - p_2). \end{aligned} \tag{69}$$

The integrals $A_{A,S}(p_1, p_2)$ are more subtle. The integral A_S is clearly regular on the diagonal. Since we are interested mostly on the singular behaviour of the matrix elements, we can approximate $A_S \sim 0$. On the other hand A_A is exactly zero on the diagonal. Close to the diagonal, the integral is dominated by the region where the argument of the trigonometric function is small, and thereby the integrand is non-oscillatory. Estimating the upper limit of this region as $\bar{x} \sim \hbar(p_1 - p_2)^{-1}$, we get

$$A_A \sim \hbar^{-1}(p_1 - p_2)\bar{x}^2 + (\delta_1 - \delta_2)\bar{x} = \hbar PV \left(\frac{1}{p_1 - p_2} \right) + \dots, \tag{70}$$

where the dots stand for regular terms. Actually, this argument would allow us to introduce an undetermined coefficient in front of the principal value PV , but in the next section we show that \hbar is the correct coefficient, as follows from the canonical commutation relations.

Thus, we are now in the position to give the explicit expressions for the singular parts of the matrix elements and write, finally, the quantum transport equation in its explicit form. This is done in detail in the next section.

5. The quantum transport equation

In this section, we explicitly compute the quantum transport equation (42) satisfied by the reduced Wigner function in the energy representation.

5.1. Matrix elements

First, we need to compute the matrix elements described in section 4.3. We begin with the matrix element $X_{E_1 E_2}$ which according to (66) and (69)–(70) can be written as

$$X_{E_1 E_2} = \frac{M\hbar}{\sqrt{p_1 p_2}} \left[\frac{1}{\pi} \frac{\partial}{\partial p_1} PV \left(\frac{1}{p_1 - p_2} \right) - \frac{\partial \delta_1}{\partial p_1} \delta(p_1 - p_2) + \dots \right]. \tag{71}$$

We go next to the matrix element $P_{E_1 E_2}$, which from (68) and (70) can be written as

$$P_{E_1 E_2} = \frac{-iM}{\sqrt{p_1 p_2}} \frac{1}{2\pi} (p_1 + p_2) PV \frac{1}{p_1 - p_2}. \tag{72}$$

These two operators X and P are connected through equation (60). It is easy to check that the two previous results satisfy this relation. Just note that from equation (A.17) we can write $E_1 - E_2 = (p_1^2 - p_2^2)/2M$ which together with equation (71) for $X_{E_1 E_2}$ lead to $-i\hbar/M$ times the right-hand side of equation (72), that is,

$$(E_1 - E_2)X_{E_1 E_2} = -\frac{i\hbar}{M} P_{E_1 E_2}.$$

Another check of the previous results is the consistency with the canonical commutation relations

$$\int dE (P_{E_1 E} X_{E E_2} - X_{E_1 E} P_{E E_2}) = -i\hbar \delta(E_1 - E_2). \quad (73)$$

This check requires a little more work. First, it is convenient to change to momentum variables and write $\delta(E_1 - E_2) = (M/\sqrt{p_1 p_2})\delta(p_1 - p_2)$. Then one needs to compute the integral

$$I \equiv \hbar \int_{-\infty}^{\infty} dp PV \left(\frac{1}{p_1 - p} \right) PV \left(\frac{1}{p - p_2} \right) = -\hbar \pi^2 \delta(p_1 - p_2). \quad (74)$$

The evaluation of this integral is easily performed using the following representation of the principal value,

$$PV \left(\frac{1}{p} \right) = \int \frac{d\xi}{2\pi\hbar} e^{ip\xi/\hbar} (-i\pi \text{sign}[\xi]),$$

which is easily proved by taking the Fourier transform of $PV(1/p)$. With the result of equation (74) it is straightforward to check that the commutation relation (73) is an identity within our approximation. This consistency check is important because it can be used to fix to \hbar the coefficient in front of the principal value of A_A in the argument leading to equation (70).

We can now move to the matrix elements for X^2 . Having an expression for $X_{E_1 E_2}$ in equation (71) it is best to compute $X_{E_1 E_2}^2$ directly from the relation (61) which leads to

$$X_{E_1 E_2}^2 = \frac{M\hbar^2}{\sqrt{p_1 p_2}} \left[\frac{\partial^2}{\partial p_1 \partial p_2} \delta(p_1 - p_2) + \frac{1}{\pi} \left(\frac{\partial \delta_1}{\partial p_1} + \frac{\partial \delta_2}{\partial p_2} \right) \frac{\partial}{\partial p_2} PV \left(\frac{1}{p_1 - p_2} \right) + \left(\frac{\partial \delta_1}{\partial p_1} \right)^2 \delta(p_1 - p_2) + \dots \right], \quad (75)$$

where we have used the result (74) and performed the E integration or, more precisely, the p integration.

The matrix element $(XP)_{E_1 E_2} = \int dE X_{E_1 E} P_{E E_2}$ can be analogously obtained from expressions (71) and (72). The result is

$$(XP)_{E_1 E_2} = \frac{iM\hbar}{2\sqrt{p_1 p_2}} \left[2p_2 \frac{\partial}{\partial p_1} \delta(p_1 - p_2) + \frac{1}{\pi} \frac{\partial \delta_1}{\partial p_1} (p_1 + p_2) PV \left(\frac{1}{p_1 - p_2} \right) + \dots \right]. \quad (76)$$

A further consistency check of these expressions comes from the property (62), which is satisfied within our approximation.

5.2. The quantum transport equation and time scales

Finally, we can write the quantum transport equation (50) in a more explicit form. The coefficient Q is given by (52), with the values of the dissipation and diffusion parts given by (57)–(59), which can be directly computed using the matrix elements obtained in the previous subsection. It is convenient to introduce new Wigner function coefficients,

$$C_{E_1 E_2}(t) = \frac{M}{\sqrt{p_1 p_2}} C_{p_1 p_2}(t), \quad (77)$$

and the result is the rather cumbersome expression (C.1) given in appendix C. As explained there, we can get a local approximation of the quantum transport equation (C.1),

$$\begin{aligned} \frac{\partial C(P, p, t)}{\partial t} = & \left(\frac{-iPP}{M\hbar} + \gamma \frac{\partial}{\partial P} P + \gamma M \sigma^2 \frac{\partial^2}{\partial P^2} + i\Delta p \frac{\partial}{\partial P} \right) C(P, p, t) \\ & - \gamma M \sigma^2 \left(\frac{\partial(\delta_1 - \delta_2)}{2\partial P} + \frac{\partial(\delta_1 + \delta_2)}{\partial p} \right)^2 C(P, p, t), \end{aligned} \quad (78)$$

where the average and difference momentum variables P and p , defined in equation (C.3), have been used and where $C(P, p, t) = C_{p_1 p_2}(t)$. From the quantum transport equation (C.1), or its local version equation (78), it is easy to discuss the different time scales of the problem. The first, of course, involves the dissipation term which includes the dissipation coefficient γ ; it defines a time scale $\tau_R \sim \gamma^{-1}$ which is the relaxation time.

But before we go on with the interpretation of the different terms, it is important to recall the meaning of the Wigner function coefficients $C_{p_1 p_2}$, or $C_{E_1 E_2}$. First, we note that these coefficients are directly related to the coefficients C_E of the energy eigenfunctions which make the tunnelling state from the false vacuum in the isolated system, i.e. when there is no interaction with the environment. Thus, the coefficients $C_{E_1 E_2}$ describe the quantum correlations between the energy eigenfunctions that make the tunnelling system. These coefficients are initially separable $C_{E_1 E_2}(0) = C_{E_1}(0)C_{E_2}^*(0)$. In the isolated closed system its time evolution, as given by equation (36), is simply $C_{E_1 E_2}(t) = C_{E_1 E_2}(0) \exp[-i(E_1 - E_2)t/\hbar]$, which means that these correlations keep their amplitude in its dynamical evolution.

This is very different in the open quantum system as a consequence of the negative local normal diffusion term in equation (C.1) which depends on the phase shift derivatives, i.e. the last term of equation (78), or equivalently the term (C.2) when it is written in the p_1 and p_2 variables. This negative defined term has no effect for the diagonal coefficients, when $E_1 = E_2$, but its effect is very important for the off-diagonal coefficients. In fact, the amplitude of the off-diagonal coefficients exponentially decays in time. The time scale can be estimated by taking the derivatives of the phase shifts δ_i ($i = 1, 2$) near the false vacuum energy E_0 , which is where the energy wavefunctions pile up. Using equation (12) it is easy to see from expression (C.2) that this time scale is of the order of

$$\tau_D \sim \tau_R \left(\frac{\lambda_B}{l_D} \right)^2, \quad (79)$$

where τ_R is the relaxation time, $\lambda_B = \hbar/(2\sigma\sqrt{M})$ is a characteristic de Broglie wavelength (in the high temperature case when $\sigma^2 = k_B T$ it corresponds to the thermal de Broglie wavelength), and $l_D \sim \alpha^2 \hbar \sqrt{E_0 + U_\infty}/(\varepsilon\sqrt{M})$ is a characteristic length of the problem with α a dimensionless parameter that measures the scale of the energy differences of the off-diagonal coefficient, $E_1 - E_2 \sim \alpha\varepsilon$; so it is of order 1 when the energy differences are of order ε . Thus, the last term of equation (78) destroys the quantum correlations of the energy eigenfunctions. The time scale τ_D may be considered as a decoherence time [38], and thus the effect on tunnelling of this term may be associated with the effect of decoherence.

Another time scale in the problem is, of course, the tunnelling time which according to (26) and (9) is given by $\tau_{\text{tunn}} \sim \hbar/\varepsilon$. Its relation to τ_D is given by $\tau_D \sim \tau_{\text{tunn}}/(\alpha^4 D)$, where the dimensionless parameter D is

$$D = \gamma \hbar \sigma^2 \frac{(E_0 + U_\infty)}{\varepsilon^3}. \quad (80)$$

It seems clear that when $\tau_D \ll \tau_{\text{tunn}}$ the coefficients $C_{E_1 E_2}$ become diagonal very fast and the local approximation to the transport equation (78) is a useful approximation.

6. Tunnelling in the open quantum system

We can now compute the tunnelling rate from the false vacuum for our open quantum system. Thus, let us assume that our particle at $t = 0$ is trapped into the well of the potential (2) in the false ground state with the energy E_0 , i.e. the ground state of the auxiliary potential U_{aux} introduced in section 2.3. We know from that section that the wavefunction ψ_0 of this state

can be expressed in terms of the eigenfunctions ψ_E by equation (13) with the coefficients C_E given by equation (22).

Under the assumption that the decoherence time is much shorter than the tunnelling time the quantum transport equation (C.1) may be simplified. After a typical decoherence time the Wigner function coefficients $C_{E_1 E_2}$ become diagonal. We will begin our leading-order approximation assuming that these coefficients are diagonal, which means that decoherence is almost instantaneous, and then we will correct it in a perturbative way.

6.1. The Kramers limit

We are interested in the regime where the decoherence time is much shorter than the tunnelling time, or $D \gg 1$, i.e. when the decoherence term (C.2) suppresses the non-diagonal Wigner function coefficients $C_{p_1 p_2}(t)$ and get a totally decohered Wigner function. In this case using the average and difference momentum variables P and p , see equations (C.3), we may approximate these coefficients as $C_{p_1 p_2}(t) \equiv C(P, p, t) \sim f(P, t)\delta(p)$. The quantum transport equation (C.1), or its local approximation equation (78), then reduces to

$$\frac{\partial f}{\partial t} = \gamma \frac{\partial}{\partial P} \left(P + M\sigma^2 \frac{\partial}{\partial P} \right) f, \quad (81)$$

which is the Kramers [17] thermal activation equation for $f(P, t)$. This is a continuity equation for a distribution f with a flux $\Phi = -\gamma(P + M\sigma^2 \frac{\partial}{\partial P})f$. The computation of the tunnelling amplitude has thus been reduced to the computation of the escape probability of a particle confined to a potential $U(x)$ defined in equation (2) subject to a damping force γP and white noise with amplitude $\gamma M\sigma^2$; which corresponds to a temperature $T = \sigma^2/k_B$. The boundary conditions of the Kramers problem are the usual ones. We assume that the particle is initially trapped at the potential well. Let us define $P_0 = \sqrt{2MU_\infty}$; we may take $U_\infty = 0$ in this section since the asymptotic value of the potential outside the barrier has no relevance for the classical activation problem. We will ask that there is no flux entering into the well so that $\Phi(0) = 0$ and $\partial f/\partial P(0) = 0$. Moreover, at the separatrix when $P_s = \sqrt{2M\varepsilon_s}$, where ε_s is the potential barrier, $f(P_s) = 0$. This means that the number of particles above the separatrix is negligible.

To find the activation rate we seek normal modes $f = f(P)e^{-rt}$ and assume that r is very small. If $r = 0$ we have the stationary solution $f_0 = \exp(-P^2/2M\sigma^2)$ which satisfies the boundary condition at $P = 0$ but not at P_s . We seek a second stationary solution, for $r = 0$, of the form $F_0 = f_0 f_s$, and then equation (81) reduces to

$$\gamma M\sigma^2 \frac{\partial}{\partial P} \left(f_0 \frac{\partial}{\partial P} f_s \right) = 0, \quad (82)$$

which leads to $\partial f_s/\partial P = K/f_0$, where the value of the constant K is irrelevant, and may be chosen as $K = -1$. The resulting solution $F_0(P) = f_0 \int_P^{P_s} dQ f_0^{-1}(Q)$ satisfies the boundary condition at P_s but not at $P = 0$. Thus, we now have two stationary solutions of equation (81), f_0 and F_0 , and we may use the variation of constants method to find a normal mode solution for $r > 0$. Let us write

$$f = \alpha(P)f_0 + \beta(P)F_0, \quad (83)$$

with the supplementary condition $\alpha' f_0 + \beta' F_0 = 0$. The function $\alpha(P)$ and $\beta(P)$ are determined by substitution into equation (81) which leads to the first-order differential equations $\alpha' = -(r/\gamma M\sigma^2)(\alpha F_0 + \beta F_0^2/f_0)$ and $\beta' = (r/\gamma M\sigma^2)(\alpha f_0 + \beta F_0)$; the boundary

conditions are $\beta(0) = \alpha(P_s) = 0$. For r very small, we may just approximate $\alpha = \alpha(0)$ and $\beta = 0$ on the right-hand sides of those equations and we get

$$\alpha(P) = \alpha(0) \left(1 - \frac{r}{\gamma M \sigma^2} \int_0^P dQ F_0(Q) \right), \quad (84)$$

which leads to the lowest eigenvalue

$$r = \frac{\gamma M E_0}{\int_0^{P_s} dQ F_0(Q)} \sim \frac{\gamma}{\sqrt{\pi}} \sqrt{\frac{\varepsilon_s}{\sigma^2}} e^{-\varepsilon_s/\sigma^2}. \quad (85)$$

This is the escape rate of the particle. We may give this rate in terms of the effective escape temperature defined in equation (41) by equating r to $(1/2\tau) \exp(-\varepsilon_s/k_B T_{\text{esc}})$, that is,

$$T_{\text{esc}} = \frac{\sigma^2}{k_B} \left[1 - \frac{\sigma^2}{\varepsilon_s} \ln \left(\frac{2\gamma}{\Omega_0} \sqrt{\frac{\pi \varepsilon_s}{\sigma^2}} \right) \right]^{-1}, \quad (86)$$

where we have used that the dynamical time τ , defined in equation (8), is $\tau \sim \pi/\Omega_0$. Thus, we have an escape temperature of the order of $T_{\text{esc}} \sim \sigma^2/k_B \leq \frac{1}{2} \hbar \Omega_0/k_B$, at zero temperature.

6.2. Effect of anomalous diffusion

We may now estimate the effect of anomalous diffusion on the tunnelling rate. We have seen in section 4 that the effect of anomalous diffusion on the linear entropy of the reduced density matrix of the open quantum system is undefined, unlike the effect of normal diffusion. It may increase or decrease the entropy, and hence the decoherence, depending on the product of gradients of the Wigner function on phase space; see equation (48). Let us go back to the quantum transport equation (50). The term $Q^{(N)}$ is dominated by the decoherence term (C.2); for this reason we may distinguish a fast and a slow dynamics. The fast dynamics corresponds to the decay of the non-diagonal Wigner function coefficients. The slow dynamics is the diffusion of the Wigner function coefficients along the diagonal which will be considered in the next section and may be described by a Fokker–Planck equation for a classical distribution function.

To evaluate the effect of $Q^{(A)}$, let us consider Wigner function coefficients $C_{E_1 E_2}$ of the form

$$C = C^{\text{diag}} + C^{\text{odd}}, \quad (87)$$

where C^{diag} is diagonal and C^{odd} is antisymmetric. Here we do not include symmetric off-diagonal coefficients, such as C^{even} , because we assume that decoherence takes them to zero, even though other terms generate it. According to the parity properties of the $Q^{(D)}$, $Q^{(N)}$ and $Q^{(A)}$ defined in equations (57)–(59), we may write equation (50) as a set of coupled equations:

$$\begin{aligned} \frac{\partial}{\partial t} C^{\text{diag}} &= (Q^{(D)} + Q^{(N)}) C^{\text{diag}} + Q^{(A)} C^{\text{odd}}, \\ \frac{\partial}{\partial t} C^{\text{odd}} &= (Q^{(D)} + Q^{(N)}) C^{\text{odd}} + Q^{(A)} C^{\text{diag}}. \end{aligned} \quad (88)$$

Note that the first term of equation (50) does not contribute to this set of equations: $Q^{(Q)}$ does not contribute to the first equation because it vanishes on the diagonal, nor to the second equation because it destroys diagonal Wigner functions. Note that the symmetric off-diagonal coefficients C^{even} do not couple to the diagonal coefficients C^{diag} through terms that contain a $\delta(p)$, thus these terms are second order with respect to the diagonal coefficients as analysed in section 6.1. Because the leading process is decoherence, the second equation may be

approximated by $\frac{\partial}{\partial t} C^{\text{odd}} = \frac{-1}{\tau_D} C^{\text{odd}} + Q^{(A)} C^{\text{diag}}$ where τ_D is a typical decoherence time scale estimated in section 5.2; see equation (79). Note that this is analogous to the relaxation time approximation for the Boltzmann equation. If τ_D is short enough, C^{odd} simply trails C^{diag} and $C^{\text{odd}} \sim \tau_D Q^{(A)} C^{\text{diag}}$ and we obtain an autonomous equation for C^{diag} ,

$$\frac{\partial}{\partial t} C^{\text{diag}} = [Q^{(D)} + Q^{(N)} + \tau_D (Q^{(A)})^2] C^{\text{diag}}. \quad (89)$$

To compute $(Q^{(A)})^2$ we go back to equation (59), we permute operators X and P so that we get an expression in terms of P and P^2 only. The linear terms in P vanish near the diagonal and then replace P with a typical momentum scale, such as $\sqrt{M\sigma^2}$. Finally, we get

$$(Q^{(A)})^2_{E_1 E_2, E'_1 E'_2} \sim \frac{4\Delta^2 M \sigma^2}{\hbar^2} (X_{E_1 E'_1}^2 \delta_{E_2 E'_2} + \delta_{E_1 E'_1} X_{E_2 E'_2}^2 - 2X_{E_1 E'_1} X_{E_2 E'_2}), \quad (90)$$

which is the same operator as $Q^{(N)}$. Thus, equation (89) leads to the first of the previous set of coupled equations when the anomalous term is neglected, but with a modified normal diffusion coefficient. The effect of anomalous diffusion is then to lower the normal diffusion coefficient σ^2 to σ_{eff}^2 , where

$$\sigma_{\text{eff}}^2 = \left(1 - \frac{4\tau_D \Delta^2}{\gamma}\right) \sigma^2. \quad (91)$$

This translates into a lowering of the effective escape temperature of equation (86), namely $T_{\text{esc}} \sim \sigma_{\text{eff}}^2/k_B$. Thus the escape temperature is always lower than σ^2/k_B but it tends to it when the decoherence time τ_D goes to zero. Consequently, the overall effect of anomalous diffusion is to inhibit tunnelling when strong decoherence is assumed.

7. Conclusions

To conclude, let us briefly summarize our results. Under the assumption of strong decoherence and using a real-time formalism we have estimated the tunnelling rate for an open quantum system representing a quantum particle, trapped in a local minimum of a cubic potential, coupled to an environment. The real-time formalism is based on the master equation for the reduced Wigner function that describes the open quantum system. Our computational method involves the introduction of an energy representation of the reduced Wigner function which is based on the energy eigenfunctions of the isolated system. The master equation in this representation, equation (50), is an equation for some Wigner function coefficients that describe the quantum correlations between eigenfunctions of different energies.

In spite of its apparent straightforwardness, the original master equation (42) contains a term with three derivatives of the Wigner function, which makes it quite hard to handle either analytically or numerically. From this point of view the reduction of equation (42) to equation (50), which is then approximated by equation (78), is already a definite step forward. Also, the basic process of decoherence is hard to discern from equation (42), while it is clearly depicted in equation (78). A striking feature of the decay process is that in last analysis it is robust against the details of the potential barrier. For example, the Kramers activation rate is dominated by a single feature of the barrier (its height), and similarly the quantum tunnelling rate for the closed system. This simplicity is lost in equation (42) but regained in equation (78). We see that the details of the potential (such as the height of the barrier) enter the problem only through the boundary conditions and the phase shifts in the energy eigenfunctions. In particular, the strength of decoherence is determined by the phase shifts near the false vacuum energy, where the simple approximation of equation (12) applies.

In our problem, where the particle is initially trapped in the false vacuum, the master equation involves a term (C.2) that destroys the quantum correlations of the eigenfunctions and is, thus, responsible for decoherence. The strength of this term is characterized by the dimensionless parameter D , defined in equation (80), which is directly proportional to the energy difference between the false and true vacua. Under the assumption of strong decoherence the pure quantum channel to tunnelling is partially suppressed, since decoherence destroys the fine tuning among the energy eigenfunctions that makes tunnelling possible in the isolated closed system. Tunnelling then follows an activation-like channel due to the zero point fluctuations of the quantum environment. This is similar to the result recently found in [45] for a double-well potential, which uses a large-scale numerical simulation to solve the master equation.

Thus the picture we have is the following. For the isolated closed system, tunnelling from the ground state goes through the usual quantum channel which the WKB approach or the instanton approach reproduces, and that we can equally reproduce using the energy representation of the master equation; see section 3.2. When the system is coupled to an environment the general effect is felt as dissipation and diffusion, the latter coming as normal and anomalous diffusion. These terms will produce essentially two effects. On the one hand, they will produce decoherence to the system which will tend to suppress tunnelling as the system becomes more classical. On the other hand, diffusion will also introduce noise into the system which will induce tunnelling by a mechanism similar to thermal activation. In general, all these mechanism act simultaneously and their effects cannot be disentangled.

When decoherence is very fast quantum tunnelling is strongly suppressed and activation dominates, to leading order. In section 4 we have seen how the diffusion term, the normal diffusion and anomalous diffusion affect the linear entropy of the reduced matrix density of the open quantum system. We see clearly that normal diffusion always tends to increase the entropy and thus to induce classicalization and decoherence. This is in fact what the decoherence term (C.2) in the quantum transport equation does. The fact that activation from normal diffusion becomes the dominant decay mechanism when decoherence is fast enough validates the analysis of vacuum decay in cosmology and quantum field theory given in [25, 33, 34]. On the other hand, these references show how the analysis given here may be improved by a more realistic description of the bath and the system-environment interaction.

We should emphasize that although the master equation in the energy representation, equation (50), is much simpler than in the standard phase space representation, equation (42), it may be difficult to go beyond the present results by analytic means. It may still be possible to treat perturbatively the symmetric but non-diagonal terms in the Wigner function coefficients. These terms were neglected in section 6.2 where we considered the effect of anomalous diffusion assuming coefficients of the form (87). The quantum coefficient $Q^{(Q)}$, which induces pure quantum tunnelling in the isolated system, will have an imprint in those terms.

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Appendix A. WKB solution

In this appendix we solve the WKB problem posed in section 2.2. The starting point are equations (3)–(5) with the cubic potential of equation (2). We have to match the WKB solutions in the different regions across the potential function.

A.1. Matching from forbidden to allowed regions

Let x_0 be a classical turning point $U(x_0) = E$, and let $U'(x_0) < 0$. Then to the left of x_0 we have a forbidden region, and the two corresponding independent WKB solutions of the Schrödinger equation (3) are

$$F_{\pm}(x_0, x) = \frac{e^{\pm S(x_0, x)/\hbar}}{\sqrt{2p(x)/\hbar}}, \quad (\text{A.1})$$

whereas to the right of x_0 the two independent solutions are

$$G_{\pm}(x, x_0) = \frac{e^{\pm iS(x, x_0)/\hbar}}{\sqrt{2p(x)/\hbar}}, \quad (\text{A.2})$$

and we wish to find the corresponding matching conditions. For $x \rightarrow x_0^-$, we can Taylor-expand the potential around x_0 and write $p(x) = \kappa(x_0 - x)^{1/2}$ and $S(x_0, x) = \frac{2}{3}\kappa(x_0 - x)^{3/2}$, where we have introduced $\kappa = \sqrt{2M|U'(x_0)|}$. Similarly for $x \rightarrow x_0^+$, we have $p(x) = \kappa(x - x_0)^{1/2}$ and $S(x, x_0) = \frac{2}{3}\kappa(x - x_0)^{3/2}$.

If we write $x - x_0 = e^{i\pi}(x_0 - x)$ then $iS(x, x_0) = S(x_0, x)$ and it would seem that simple analytical continuation yields $G_+(x, x_0) \rightarrow e^{-i\pi/4}F_+(x_0, x)$. However, this is impossible; recall that if we define the flux $J = -i(\psi^*\partial_x\psi - \psi\partial_x\psi^*)$ then the Schrödinger equation implies flux conservation $\partial_x J = 0$. Now $G_+(x, x_0)$ has $J = 1$ and therefore it cannot turn into $F_+(x_0, x)$, which is real, and has $J = 0$. Thus, we try instead

$$G_+(x, x_0) \rightarrow e^{-i\pi/4}F_+(x_0, x) + \beta F_-(x_0, x), \quad (\text{A.3})$$

and imposing flux conservation we obtain $\beta = (1/2)\exp(i\pi/4)$. We therefore find the matching conditions

$$e^{\mp i\pi/4}F_+(x_0, x) + \frac{1}{2}e^{\pm i\pi/4}F_-(x_0, x) \rightarrow G_{\pm}(x, x_0), \quad (\text{A.4})$$

from where we finally obtain, using equation (A.2),

$$F_+(x_0, x) \rightarrow \frac{1}{\sqrt{2p(x)/\hbar}} \cos\left(\frac{1}{\hbar}S(x, x_0) + \frac{\pi}{4}\right), \quad (\text{A.5})$$

and

$$F_-(x_0, x) \rightarrow \frac{2}{\sqrt{2p(x)/\hbar}} \sin\left(\frac{1}{\hbar}S(x, x_0) + \frac{\pi}{4}\right). \quad (\text{A.6})$$

A.2. Matching from allowed to forbidden regions

Now consider the case when $U'(x_0) > 0$. To the left of x_0 , we have an allowed region and the solutions are oscillatory $G_{\pm}(x_0, x)$, and to the right of the turning point we have a forbidden region and the solutions are a linear combination of (A.1). By exactly the same procedure as in the previous section, after imposing flux conservation across x_0 we obtain

$$G_{\mp}(x_0, x) \rightarrow e^{\pm i\pi/4}F_+(x, x_0) + \frac{1}{2}e^{\mp i\pi/4}F_-(x, x_0). \quad (\text{A.7})$$

Note from these equations that the solution that matches a decreasing exponential is

$$\frac{1}{\sqrt{2p(x)/\hbar}} \sin\left(\frac{1}{\hbar}S(x_0, x) + \frac{\pi}{4}\right) \rightarrow \frac{1}{2}F_-(x, x_0). \quad (\text{A.8})$$

A.3. WKB solution for $0 < E < \varepsilon_s$

We can now put all this together to write the energy eigenfunctions for our cubic potential (2) for energies in the range $0 < E < \varepsilon_s$. There are three classical turning points in this case $x_L < x_R < x_{\text{out}}$. To the left of x_L we have a forbidden zone extending to $-\infty$, so we have

$$\psi_E(x) \sim K_E F_-(x_L, x), \quad x < x_L, \quad (\text{A.9})$$

where K_E is a normalization constant to be determined later. To the right of x_L we have from (A.6)

$$\psi_E(x) \sim \frac{2K_E}{\sqrt{2p(x)/\hbar}} \sin\left(\frac{1}{\hbar}S(x, x_L) + \frac{\pi}{4}\right), \quad (\text{A.10})$$

which after using the definition (5) can be rewritten in the region $x_L < x < x_R$ as

$$\psi_E(x) \sim K_E (e^{i(S(x_R, x_L)/\hbar - \pi/4)} G_-(x_R, x) + e^{-i(S(x_R, x_L)/\hbar - \pi/4)} G_+(x_R, x)). \quad (\text{A.11})$$

This expression is in the form suitable for extension to the forbidden region, that is, to the right of x_R . Thus, by using (A.7) we have to the right of x_R

$$\psi_E(x) \sim 2K_E \left[\cos\left(\frac{1}{\hbar}S(x_R, x_L)\right) F_+(x, x_R) + \frac{1}{2} \sin\left(\frac{1}{\hbar}S(x_R, x_L)\right) F_-(x, x_R) \right], \quad (\text{A.12})$$

which can be rewritten again as

$$\begin{aligned} \psi_E(x) \sim 2K_E \left[\cos\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{S(x_{\text{out}}, x_R)/\hbar} F_-(x_{\text{out}}, x) \right. \\ \left. + \frac{1}{2} \sin\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{-S(x_{\text{out}}, x_R)/\hbar} F_+(x_{\text{out}}, x) \right], \end{aligned} \quad (\text{A.13})$$

which is in a form suitable for extension to the right of x_{out} :

$$\begin{aligned} \psi_E(x) \sim \frac{2K_E}{\sqrt{2p(x)/\hbar}} \left[2 \cos\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{S(x_{\text{out}}, x_R)/\hbar} \sin\left(\frac{1}{\hbar}S(x, x_{\text{out}}) + \frac{\pi}{4}\right) \right. \\ \left. + \frac{1}{2} \sin\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{-S(x_{\text{out}}, x_R)/\hbar} \cos\left(\frac{1}{\hbar}S(x, x_{\text{out}}) + \frac{\pi}{4}\right) \right]. \end{aligned} \quad (\text{A.14})$$

Note that if we impose the Bohr–Sommerfeld quantization rule

$$S(x_R, x_L) = \frac{\pi}{2}(1 + 2n)\hbar, \quad (\text{A.15})$$

only the subdominant, exponential decreasing part survives. This would correspond to the case when the far right region is forbidden and may be used to define energies for false states trapped into the potential well; in particular $n = 0$ will correspond to the false ground state.

A.4. Normalization

All that remains now is the determination of the normalization constant K_E which can be done from the normalization of the wavefunctions. The eigenfunctions are subject to continuous normalization

$$\int dx \psi_{E_1}(x) \psi_{E_2}(x) = \delta(E_1 - E_2). \quad (\text{A.16})$$

Since the functions themselves are regular, the singular behaviour must come from the upper limit; see, for instance, [47]. For large enough x , we have from equation (5)

$$p \rightarrow p_\infty = \sqrt{2M(E + U_\infty)}. \quad (\text{A.17})$$

Let us write from equation (A.1),

$$S(x, x_{\text{out}}) = p_{\infty}(x - x_{\text{out}}) + \int_{x_{\text{out}}}^x dx' [\sqrt{2M(E - U(x'))} - \sqrt{2M(E + U_{\infty})}], \quad (\text{A.18})$$

if this integral converges, and we may take the upper limit of integration to ∞ , whereby

$$S(x, x_{\text{out}}) = p_{\infty}x + f(E), \quad (\text{A.19})$$

where $f(E)$ stands for the second term of (A.18). Then, for $x \gg x_{\text{out}}$, we can write from (A.14) and (A.19)

$$\psi_E(x) \sim \frac{\sqrt{2\hbar}K_E}{\sqrt{p_{\infty}}} \left[A(E) \sin\left(\frac{p_{\infty}x}{\hbar}\right) + B(E) \cos\left(\frac{p_{\infty}x}{\hbar}\right) \right], \quad (\text{A.20})$$

where $A(E)$ and $B(E)$ are given by

$$\begin{aligned} A(E) &= 2 \cos\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{S(x_{\text{out}}, x_R)/\hbar} \cos\left(f(E) + \frac{\pi}{4}\right) \\ &\quad - \frac{1}{2} \sin\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{-S(x_{\text{out}}, x_R)/\hbar} \sin\left(f(E) + \frac{\pi}{4}\right), \end{aligned} \quad (\text{A.21})$$

$$\begin{aligned} B(E) &= 2 \cos\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{S(x_{\text{out}}, x_R)/\hbar} \sin\left(f(E) + \frac{\pi}{4}\right) \\ &\quad + \frac{1}{2} \sin\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{-S(x_{\text{out}}, x_R)/\hbar} \cos\left(f(E) + \frac{\pi}{4}\right). \end{aligned} \quad (\text{A.22})$$

Substituting equation (A.20) into (A.16), the singular terms in the normalization integral are

$$\int dx \psi_{E_1}(x) \psi_{E_2}(x) \sim \hbar^2 \pi \frac{K_{E_1}^2}{p_1} [A^2(E_1) + B^2(E_1)] \left[\frac{dp_1}{dE_1} \right]^{-1} \delta(E_1 - E_2),$$

where the delta function comes from the x integration which brings $\delta(p_1 - p_2)$ and where we have defined $p_i \equiv p_{\infty}(E_i)$ ($i = 1, 2$) and changed from momentum to energy variables according to $p_i dp_i = M dE_i$; see equation (A.17). The normalization condition reduces to 1, the coefficient of the delta function above

$$\hbar^2 \pi \frac{K_{E_1}^2}{M} [A^2(E_1) + B^2(E_1)] = 1. \quad (\text{A.23})$$

This suggests the introduction of the phase δ_E as follows:

$$K_E A(E) = \sqrt{\frac{M}{\hbar^2 \pi}} \cos \delta_E, \quad K_E B(E) = \sqrt{\frac{M}{\hbar^2 \pi}} \sin \delta_E. \quad (\text{A.24})$$

Thus, the eigenfunction at $x \gg x_{\text{out}}$ is equation (6), that is,

$$\psi_E(x) \sim \sqrt{\frac{2M}{\hbar \pi p_{\infty}}} \sin\left(\frac{p_{\infty}x}{\hbar} + \delta_E\right).$$

To work out the constant K_E in greater detail, we note that from equations (A.21) and (A.22) we have

$$A^2 + B^2 = 4 \cos^2\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{2S(x_{\text{out}}, x_R)/\hbar} + \frac{1}{4} \sin^2\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{-2S(x_{\text{out}}, x_R)/\hbar}, \quad (\text{A.25})$$

which is non-vanishing as long as E is real. However, if we allow for complex energies, as is typical of unstable states, it may be zero, provided

$$\cos^2\left(\frac{1}{\hbar}S(x_R, x_L)\right) = \frac{-1}{16} \sin^2\left(\frac{1}{\hbar}S(x_R, x_L)\right) e^{-4S(x_{\text{out}}, x_R)/\hbar}. \quad (\text{A.26})$$

The left-hand side is zero whenever the energy satisfies the Bohr–Sommerfeld condition (A.15).

Appendix B. Thermal spectrum

In this appendix we check that the quantum transport equation (50) admits a stationary solution with a thermal spectrum. This can be seen as a test on the restrictions satisfied by the matrix elements (53)–(56) with $\sigma^2 = k_B T$.

An unnormalized thermal density matrix in the position representation reads

$$\rho(x, x') = \int dE e^{-\beta E} \psi_E(x) \psi_E(x'), \quad (\text{B.1})$$

where $\beta = (k_B T)^{-1}$ and its associated Wigner function is

$$W_\beta(x, p) = \int dE e^{-\beta E} W_{EE}(x, p), \quad (\text{B.2})$$

which in the energy representation in the base $W_{E_1 E_2}$ of equation (35) corresponds to the coefficients $C_{E_1 E_2} = e^{-\beta E_1} \delta(E_1 - E_2)$. Inserting this into the transport equation we get

$$\int dE e^{-\beta E} Q_{E_1 E_2, EE} = 0, \quad (\text{B.3})$$

which after using equations (52), (57) and (58) can be written in operator language as

$$0 = \frac{1}{2M} \frac{i}{\hbar} (XP e^{-\beta H} - e^{-\beta H} PX - P e^{-\beta H} X + X e^{-\beta H} P) \\ + \frac{1}{\beta \hbar^2} (X^2 e^{-\beta H} + e^{-\beta H} X^2 - 2X e^{-\beta H} X).$$

At the infinite temperature limit, $\beta = 0$, this is

$$0 = \frac{1}{M} \frac{i}{\hbar} (XP - PX) - \frac{1}{\hbar^2} (X^2 H + H X^2 - 2X H X). \quad (\text{B.4})$$

The first term is the commutator which gives $-M^{-1}$, and the second term can be written as $-\frac{1}{\hbar^2} [X, [X, H]]$, which using $[H, X] = (\hbar/i)(P/M)$ is easily seen to cancel the first term.

Appendix C. Quantum transport equation

Here we write explicitly the quantum transport equation (50) in the energy representation. The coefficient Q in equation (50) is given by (52), and the values of the dissipative, normal diffusion and anomalous diffusion parts of this coefficient are given, respectively, by equations (57), (58) and (59). These parts can be directly written using the matrix elements deduced in section 5. When the coefficients $C_{p_1 p_2}$ defined in equation (77) are introduced, the transport equation becomes

$$\frac{\partial C_{p_1 p_2}}{\partial t} = \frac{-i}{2M\hbar} (p_1^2 - p_2^2) C_{p_1 p_2} + \gamma M \sigma^2 \left(\frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} \right) C_{p_1 p_2} \\ + \frac{\gamma}{2} \left(\frac{\partial}{\partial p_1} p_1 + \frac{\partial}{\partial p_2} p_2 \right) C_{p_1 p_2} + i\Delta \left(p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2} \right) C_{p_1 p_2} \\ + \frac{\gamma - i2\Delta}{4\pi^2} \frac{\partial}{\partial p_1} \int dp'_1 dp'_2 (p_2 + p'_2) P(p_2 - p'_2) P(p_1 - p'_1) C_{p'_1 p'_2} \\ + \frac{\gamma + i2\Delta}{4\pi^2} \frac{\partial}{\partial p_2} \int dp'_1 dp'_2 (p_1 + p'_1) P(p_2 - p'_2) P(p_1 - p'_1) C_{p'_1 p'_2} \\ + \frac{2\gamma M \sigma^2}{\pi^2} \frac{\partial^2}{\partial p_1 \partial p_2} \int dp'_1 dp'_2 P(p_1 - p'_1) P(p_2 - p'_2) C_{p'_1 p'_2}$$

$$\begin{aligned}
& -\gamma M\sigma^2 \left(\frac{\partial\delta_1}{\partial p_1} - \frac{\partial\delta_2}{\partial p_2} \right)^2 C_{p_1 p_2} \\
& + \frac{\gamma}{4\pi} \left(\frac{\partial\delta_1}{\partial p_1} - \frac{\partial\delta_2}{\partial p_2} \right) \int dp [(p_1 + p)P(p_1 - p)C_{pp_2} - (p_2 + p)P(p_2 - p)C_{p_1 p}] \\
& + \frac{i\Delta}{2\pi} \left(\frac{\partial\delta_1}{\partial p_1} - \frac{\partial\delta_2}{\partial p_2} \right) \int dp [(p_1 + p)P(p_1 - p)C_{pp_2} + (p_2 + p)P(p_2 - p)C_{p_1 p}] \\
& + \gamma M\sigma^2 \int \frac{dp'_2}{\pi} \left(\frac{\partial\delta_2}{\partial p_2} + \frac{\partial\delta_2'}{\partial p'_2} - 2\frac{\partial\delta_1}{\partial p_1} \right) \frac{\partial P(p_2 - p'_2)}{\partial p_2} C_{p_1 p'_2} \\
& + \gamma M\sigma^2 \int \frac{dp'_1}{\pi} \left(\frac{\partial\delta_1}{\partial p_1} + \frac{\partial\delta_1'}{\partial p'_1} - 2\frac{\partial\delta_2}{\partial p_2} \right) \frac{\partial P(p_1 - p'_1)}{\partial p_1} C_{p'_1 p_2}, \tag{C.1}
\end{aligned}$$

where we have used the shorthand notation $P(x) \equiv PV(1/x)$.

This equation may be considerably simplified by noting the effect of the negative defined local term which depends on the phase shift derivatives:

$$-\gamma M\sigma^2 \left(\frac{\partial\delta_1}{\partial p_1} - \frac{\partial\delta_2}{\partial p_2} \right)^2 C_{p_1 p_2}. \tag{C.2}$$

This term has no effect on the diagonal Wigner function coefficients, when $p_1 = p_2$, but it exponentially reduces the off-diagonal coefficients $C_{p_1 p_2}$ on a time scale of the decoherence time, τ_D , as discussed in section 5. This suggests the following local approximation to the transport equation (C.1) whenever $\tau_D \ll \tau_{\text{tunn}}$.

To derive the local approximation it is best to introduce new average and difference momentum variables,

$$P = \frac{1}{2}(p_1 + p_2), \quad p = p_1 - p_2, \tag{C.3}$$

then, when substituting into equation (C.1), we have two typical terms involving the p'_1 and p'_2 integrations,

$$\begin{aligned}
& \int dU du (2P + U)P(U - u)P(U + u)C(P + U, p + 2u), \\
& \int dU du (p + u)P(U - u)P(U + u)C(P + U, p + 2u),
\end{aligned}$$

where we have used $C(P, p) = C_{p_1 p_2}$, and $U = P' - P$ and $u = \frac{1}{2}(p' - p)$. Now we make the hypothesis that the P dependence is softer than the p dependence, so within these integrals we can approximate $C(P + U, p + 2u) \sim C(P, p + 2u)$. Then using equation (74) we can integrate $\int dU P(U - u)P(U + u) = \frac{\pi^2}{2}\delta(u)$ and $\int dU U P(U - u)P(U + u) = 0$. In this way, all the terms in equation (C.1) which do not depend on the phase shifts δ_1 and δ_2 become local and considerably simplified. For the phase shift terms we may note that far from the resonance all terms are negligible and close to the resonance the local term (C.2) is clearly dominant so it makes sense to keep only this term. Finally, equation (78) follows as the local approximation of the quantum transport equation (C.1).

Appendix D. Tunnelling rates for the closed system

In this appendix we review the calculation of the quantum mechanical tunnelling rate for the closed system, that is, ignoring the interaction with the environment. The quantum tunnelling

rate as given by the instanton calculation [2, 6, 67] is

$$\Gamma_{\text{closed}}^{(\text{inst})} = \frac{a_q}{2\tau} e^{-\Lambda_0}, \quad (\text{D.1})$$

where $\tau = \pi/\Omega_0$, $\Lambda_0 = S_B/\hbar = 18\varepsilon_s/(5\varepsilon_0) \sim 12.376$, and the prefactor $a_q = (120\pi\Lambda_0)^{1/2} \sim 68.306$; here we use as an example the values $\varepsilon_s/k_B \sim 589.74$ mK, and $\varepsilon_0/k_B \sim 171.55$ mK, from a tunnelling experiment for a single Josephson junction between two superconducting electrodes biased by an external current reported in [2]. With these values, the escape temperature defined in equation (41) is

$$T_{\text{esc}}^{(\text{inst})} = \frac{\varepsilon_0/k_B}{3.6 - (\varepsilon_0/\varepsilon_s) \ln a_q} \sim 72.345 \text{ mK}. \quad (\text{D.2})$$

It is interesting to check that this result agrees with the result we obtain when the dissipation is zero. We can use our WKB result as obtained in section 2, see equation (26), to write

$$\Gamma_{\text{closed}}^{(\text{WKB})} = \frac{1}{2\tau} e^{-\Lambda}, \quad (\text{D.3})$$

where $\Lambda = (2/\hbar)S_0(x_{\text{out}}, x_R)$, with S_0 defined in equations (4)–(5), where the potential $U(x)$ is given by equation (2).

For a cubic potential, the relationship among the energy E , the frequency Ω and the action $S(x_R, x_L)$ is best given in parametric form,

$$E = 2\varepsilon_s\zeta(k), \quad \Omega = \Omega_0 f(k), \quad S(x_R, x_L) = \frac{\varepsilon_s}{\Omega_0} F(k), \quad (\text{D.4})$$

with $0 < k < 1$, and

$$\zeta(k) = \frac{1}{8} \left\{ 2 + 3 \frac{(1+k^2)}{[Q(k)]^{1/2}} - \frac{(1+k^2)^3}{[Q(k)]^{3/2}} \right\}, \quad (\text{D.5})$$

$$f(k) = \left\{ \frac{2}{\pi} [4Q(k)]^{1/4} K[k^2] \right\}^{-1}, \quad (\text{D.6})$$

$$F(k) = \frac{27}{8} \left[\frac{4}{Q(k)} \right]^{5/4} \{a(k)E[k^2] - (1-k^2)b(k)K[k^2]\},$$

where $E[k^2]$ and $K[k^2]$ are the complete elliptic integrals, and we have introduced the functions $Q(k) = (1/4)(1+14k^2+k^4)$, $a(k) = (16/15)(2-k^2)^2 - (1/5)(1-k^2)(21-5k^2)$ and $b(k) = (8/15)(2-k^2) - (1-k^2)$.

The Bohr–Sommerfeld condition (A.15) for the ground state ($n = 0$) corresponds to the parameter k_{GS} such that $F(k_{GS}) = \pi\varepsilon_0/\varepsilon_s$ which implies that $k_{GS} \sim 0.1152$. This corresponds to $\zeta(k_{GS}) \sim 0.1423$ and $f(k_{GS}) \sim 0.9550$, while the harmonic approximation for the potential yields 0.1454 and 1, respectively.

To compute the barrier penetrability, $\Lambda = (2/\hbar)S_0(x_{\text{out}}, x_R)$, we observe that $S(x_{\text{out}}, x_R)$ at energy E is equal to $S(x_R, x_L)$ at energy $E_{\text{ref}} = \varepsilon_s - E$. The exchange of E by E_{ref} is equivalent to the exchange of k by k_{ref} , where $\zeta(k_{\text{ref}}) = 1/2 - \zeta(k)$. For k_{GS} we obtain $k_{\text{ref}} \sim 0.2433$ and $F(k_{\text{ref}}) \sim 2.4073$. Therefore

$$\Lambda = \frac{\varepsilon_s}{\varepsilon_0} F(k_{\text{ref}}) \sim 8.459. \quad (\text{D.7})$$

This is to be compared against the instanton exponent $\Lambda_0 - \ln a_q \sim 8.152$. In terms of the escape temperature, the WKB approximation yields

$$T_{\text{esc}}^{(\text{WKB})} = \frac{\varepsilon_0/k_B}{F(k_{\text{ref}}) - (\varepsilon_0/\varepsilon_s) \ln(\Omega_{GS}/\Omega_0)} \sim 70.869 \text{ mK}, \quad (\text{D.8})$$

which is in good agreement with the instanton result; here we have again used the previous numerical results for ε_s and ε_0 . This agreement, of course, should not be surprising since for a closed system our method reduces to the standard WKB calculation. The purpose of this exercise is just to check the consistency of our calculation and to illustrate how the instanton and WKB methods compare. That the difference between $\exp(\Lambda_0)$ and $\exp(\Lambda)$ is accounted for by the prefactor a_q of equation (D.1) can be seen analytically by a perturbative calculation.

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